



INVESTIGATION OF THE STRUCTURE OF A WEAK SHOCK WAVE AND THE PROPAGATION OF SMALL PERTURBATIONS IN GASEOUS MIXTURES USING BURNETT'S EQUATIONS†

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Two problems are investigated: the structure of a weak shock wave and the propagation of perturbations of small amplitude in a binary mixture of monatomic gases. In the first problem, the contributions to the hydrodynamic quantities are obtained in the second approximation with respect to the wave intensity. In the second problem, acoustic, thermal and diffusion mode are considered and it is shown that there is negative dispersion of the acoustic mode. © 1999 Elsevier Science Ltd. All rights reserved.

When investigating the structure of a shock wave in a binary mixture, only one dissipative mechanism was initially taken into account, namely, diffusion [1], and later [2] all dissipative mechanisms were considered and an analytic solution of the problem was obtained for molecular gases using the Navier–Stokes equations.

In the problem of the structure of a weak shock wave, the distribution of the hydrodynamic quantities in the wave is constructed in the form of power series in a small parameter ϵ —the wave intensity. In the papers mentioned above the first non-trivial terms in these expansions were taken into account, which, in particular, enabled an expression to be obtained for the thickness of the shock wave based on the maximum slope of the density profile. The finer properties of the shock-wave structure, namely, the asymmetry of the profiles of the hydrodynamic quantities, the distance between their centres, etc., have been investigated in recent years experimentally [3] and by a numerical solution of the hydrodynamic equations using the Chapman–Enskog method [4]. In the case of weak shock wave terms of the first order in ϵ in the expansions of the hydrodynamic quantities incorrectly describe these properties of the wave structure not only quantitatively but also qualitatively. Also, the effect of higher-order terms in ϵ on the shock-wave structure have not been investigated either in single-component gases or in gaseous mixtures.

In this paper, using the hydrodynamic equations of the Burnett approximation of the Chapman–Enskog method, we calculate the second-order terms in ϵ in the distributions of the hydrodynamic quantities in a weak shock wave. Using these we derive and investigate expressions for the asymmetry parameter of the density, velocity and temperature profiles, the distances between the centres of these profiles and the correction to the wave thickness. It is shown that the Burnett terms in the hydrodynamic equations are decisive in the expression for the asymmetry parameter. Only monatomic gases and their mixtures are considered, since only for these can Burnett terms be calculated completely and have a relatively simple form.

We also investigate the propagation of small perturbations in a binary mixture of monatomic gases using Burnett's equations for the case of arbitrary intermolecular interactions. In addition to acoustic modes, we also consider non-propagating modes (thermal and diffusion). The results are presented both for the initial and for the boundary-value problems.

We separately consider the case of a mixture with sharply differing molecular masses. It is shown that in such a mixture, containing a small content of the light component, negative dispersion occurs (a reduction in the velocity of sound as the frequency increases), unlike positive dispersion in single-component gases and “normal” mixtures, which gives the experimental fact a theoretical basis.

1. THE STRUCTURE OF A WEAK SHOCK WAVE

Formulation of the problem. The laws of conservation of mass of the mixture, of the mass of one of the components of the mixture (we will suppose, to fix our ideas, that this is the component of the lightest

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molecules, $m_1 < m_2$, where m_i is the mass of a molecule of the i th kind), of momentum and total energy of the mixture in a system of coordinates moving together with the wave in the negative direction of the z axis, have the form

$$\rho u = C_0, \quad C_0 = \rho^- u^- \quad (1.1)$$

$$\rho u c_1 + i_1 = C_1, \quad C_1 = \rho^- u^- \quad (1.2)$$

$$p + \rho u^2 - \sigma = C_3, \quad C_3 = p^- + \rho^- (u^-)^2 \quad (1.3)$$

$$\rho u \left(\frac{1}{2} u^2 + c_p T \right) - u \sigma + q = C_4, \quad C_4 = \rho^- u^- \left(\frac{1}{2} (u^-)^2 + c_p^- T^- \right) \quad (1.4)$$

$$c_i = \frac{\rho_i}{\rho}, \quad c_p = c_{p1} + c_{p2}, \quad c_{pi} = \frac{\gamma}{\gamma-1} R_i, \quad R_i = \frac{k_{B0}}{m_i}$$

Here c_i is the mass concentration of the i th component, c_p is the specific heat capacity of the mixture at constant pressure, in a monatomic gas $\gamma = 5/3$, k_{B0} is Boltzmann's constant, and u is the hydrodynamic velocity. The pressure p is related to the concentrations, the density ρ and the temperature T by the equation of state of a mixture of ideal gases

$$p = (c_1 R_1 + c_2 R_2) \rho T \quad (1.5)$$

The superscripts minus and plus denote quantities in the free stream and downstream of the wave, respectively.

The viscous stress tensor σ , the diffusion flux of the first component of the mixture i_1 and the heat flux q are given by the following relations [5]

$$\sigma = \frac{4}{3} \mu \frac{du}{dz} + \sigma_B \quad (1.6)$$

$$i_1 = -\rho D \left[\frac{dc_1}{dz} + c_1 c_2 \left(\frac{\alpha_p}{p} \frac{dp}{dz} + \frac{\alpha_T}{T} \frac{dT}{dz} \right) + i_{1B} \right] \quad (1.7)$$

$$q = -\kappa \frac{dT}{dz} + (c_{p1} - c_{p2}) T i_1 + \alpha_T \frac{p\rho}{n^2 m_1 m_2} i_1 + q_B \quad (1.8)$$

Here μ , D and κ are the coefficients of dynamic viscosity, diffusion and thermal conductivity, α_T is the thermal diffusion factor and $\alpha_p = (m_2 - m_1)n/\rho$.

The complete expressions for the Burnett contributions σ_B , i_{1B} , q_B in the case of mixtures of monatomic gases were obtained previously in [5]. In the approximation considered here we only need to retain terms that are linear in the derivatives. Then

$$\sigma_B = \beta_1 \frac{d^2 p}{dz^2} + \beta_2 \frac{d^2 T}{dz^2} + \beta_3 \frac{d^2 c_1}{dz^2}, \quad i_{1B} = \beta_4 \frac{d^2 u}{dz^2}$$

$$q_B^* = \beta_5 \frac{d^2 u}{dz^2}, \quad \beta_1 = \frac{\mu}{p} \left(\tilde{\omega}_1 \frac{\mu}{\rho} - \tilde{\omega}_2 \alpha_p y_1 y_2 D \right), \quad \beta_2 = -\tilde{\omega}_3 \frac{\mu \kappa}{p}$$

$$q_B = q_B^* + (c_{p1} - c_{p2}) T i_{1B} + \alpha_T \frac{p\rho}{n^2 m_1 m_2} i_{1B}, \quad y_i = \frac{n_i}{n}$$

$$\beta_3 = -\tilde{\omega}_2 \frac{\rho^2}{n m_1 m_2} \mu D, \quad \beta_4 = \frac{c_1 c_2}{p} \left(\tilde{\omega}_{4,1} \frac{\rho^2 T}{3 c_2 p} y_1 y_2 \kappa D \alpha_T + \tilde{\omega}_2 \rho \mu D - \frac{5}{3} \alpha m_1 m_2 n^2 D^2 \right),$$

$$\beta_5 = \tilde{\omega}_3 \frac{T}{p} \kappa \mu - \tilde{\omega}_5 \frac{\rho T^2}{p^2} \kappa^2 + y_1 y_2 \alpha_p \alpha_T D \left[\frac{5}{2} y_1 y_2 (\tilde{\omega}_{4,1} - \tilde{\omega}_{4,2}) \frac{\rho^2 T}{np} \kappa - \frac{5}{3} \frac{m_1 m_2 n^2}{\rho} D \right],$$

$$\tilde{\omega}_1 = \frac{4}{3\mu^2} \sum_{i=1}^2 \frac{\mu_i^2}{y_i}, \quad \tilde{\omega}_2 = \frac{4n}{3\rho\mu} \left(\frac{m_2 \mu_1}{y_1} - \frac{m_1 \mu_2}{y_2} \right),$$

$$\bar{\omega}_3 = \frac{8}{15\mu\kappa} \sum_{i=1}^2 \frac{\mu_i \kappa_i^*}{y_i}$$

$$\bar{\omega}_{4,j} = \frac{1}{3y_1 y_2 \alpha_T \kappa D} \sum_{i=1}^2 m_i \kappa_i^* d_{i,1}^{(j)}, \quad \bar{\omega}_5 = \frac{4n}{15\rho\kappa^2} \sum_{i=1}^2 \frac{m_i \kappa_i^{*2}}{y_i}$$

In the case of a single-component gas [7] we have for Maxwellian molecules

$$\bar{\omega}_1 = \frac{4}{3}, \quad \bar{\omega}_3 = \frac{8}{15}, \quad \bar{\omega}_5 = \frac{4}{15}$$

for solid spheres

$$\bar{\omega}_1 = 1.014 \cdot \frac{4}{3}, \quad \bar{\omega}_3 = 0.806 \cdot \frac{8}{15}, \quad \bar{\omega}_5 = 1.014 \cdot \frac{4}{15}$$

By analogy with the case of a single-component gas [6] we introduce the following dimensionless quantities

$$\bar{p}_i = \frac{C_3}{C_0^2} p_i, \quad \bar{p} = \frac{C_3}{C_0^2} p, \quad \bar{u} = \frac{C_0}{C_3} u, \quad \bar{p} = \frac{1}{C_3} p$$

$$\bar{T} = \frac{RC_0^2}{C_3^2} T, \quad R = \frac{R_1 C_1 + R_2 C_2}{C_0}, \quad \bar{R}_i = \frac{R_i}{R} \quad (1.9)$$

$$\bar{\mu} = \frac{\mu}{\mu^-}, \quad \bar{\kappa} = \frac{\kappa}{R\mu^-}, \quad \bar{D} = \frac{C_0^2}{C_3\mu^-} D, \quad \bar{z} = \frac{C_0}{\mu^-} z$$

In what follows it will be more convenient to choose the velocity u , the temperature T and the concentration c_1 as the independent hydrodynamic quantities. The density ρ and the pressure p are found from Eqs (1.1) and (1.5)

$$\rho = \frac{1}{u}, \quad p = (c_1 R_1 + c_2 R_2) \frac{T}{u} \quad (1.10)$$

Equations (1.2)–(1.4) can then be written in the form (here and henceforth we will omit the bar above dimensionless quantities)

$$c_1 - D \frac{1}{u} \left[\left(1 + \alpha_p c_1 c_2 \frac{R_1 - R_2}{c_1 R_1 + c_2 R_2} \right) \frac{dc_1}{dz} + c_1 c_2 \left(\frac{\alpha_p + \alpha_T}{T} \frac{dT}{dz} - \frac{\alpha_p}{u} \frac{du}{dz} \right) \right] + \beta_4 \frac{d^2 u}{dz^2} = c_1^- \quad (1.11)$$

$$u + p - \frac{4}{3} \mu \frac{du}{dz} - \beta_1 \frac{p}{u} \frac{d^2 u}{dz^2} + \left(\beta_2 + \beta_1 \frac{p}{T} \right) \frac{d^2 T}{dz^2} + \left(\beta_3 + \beta_1 \frac{R_1 - R_2}{u} T \right) \frac{d^2 c_1}{dz^2} = 1 \quad (1.12)$$

$$\frac{1}{2} u^2 + c_p T - \frac{4}{3} \mu u \frac{du}{dz} - D \frac{1}{u} \left[(c_{p1} - c_{p2}) T + \frac{\alpha_T p}{un^2 m_1 m_2} \right]$$

$$\left[\left(1 + \alpha_p c_1 c_2 \frac{R_1 - R_2}{c_1 R_1 + c_2 R_2} \right) \frac{dc_1}{dz} + c_1 c_2 \left(\frac{\alpha_p + \alpha_T}{T} \frac{dT}{dz} - \frac{\alpha_p}{u} \frac{du}{dz} \right) \right] +$$

$$+ \left[\beta_5 + \beta_4 (c_{p1} - c_{p2}) T + \beta_4 \frac{\alpha_T p}{un^2 m_1 m_2} - \beta_1 p \right] \frac{d^2 u}{dz^2} +$$

$$+ \left(\beta_1 \frac{p}{T} + \beta_2 \right) u \frac{d^2 T}{dz^2} + [\beta_1 (R_1 - R_2) T + \beta_3 u] \frac{d^2 c_1}{dz^2} = A, \quad A = \frac{C_0 C_4}{C_3^2} \quad (1.13)$$

The boundary conditions to system (1.11)–(1.13) have the form

$$c_1(\mp\infty) = c_1^\mp, \quad u(\mp\infty) = u^\mp, \quad T(\mp\infty) = T^\mp$$

$$\left(\frac{dc_1}{dz} = \frac{du}{dz} = \frac{dT}{dz} \right) \Big|_{z=\mp\infty} = 0 \quad (1.14)$$

The downstream and upstream quantities are related by the equations

$$c_1^+ = c_1^-, \quad u^+ = u^- - \frac{2\gamma}{\gamma+1} \frac{M^2 - 1}{\gamma M^2 + 1}, \quad T^\pm = u^\pm (1 - u^\pm)$$

$$u^- = \frac{\gamma M^2}{\gamma M^2 + 1}, \quad M = u^- \left(\frac{\rho^-}{\gamma p^-} \right)^{1/2}$$

The shock-wave intensity will be represented by the parameter [6]

$$\varepsilon = u^- - u^+ = \frac{2\gamma}{\gamma+1} \frac{M^2 - 1}{\gamma M^2 + 1}$$

Using the relations

$$u = \frac{\gamma}{\gamma+1} + \frac{1}{2} \varepsilon v, \quad c_1 = c_1^- + \varepsilon f$$

$$T = \frac{\gamma}{(\gamma+1)^2} + \frac{\gamma-1}{2(\gamma+1)} \varepsilon \tau - \frac{\varepsilon^2}{4} \quad (1.15)$$

we introduce the new variables v, τ, f . From (1.14) we have the following boundary conditions for these

$$z = \mp\infty : v = \pm 1, \quad \tau = \mp 1, \quad f = 0, \quad \frac{dv}{dz} = \frac{d\tau}{dz} = \frac{df}{dz} = 0$$

System of equations (1.11)–(1.13) is autonomous, i.e. it is invariant under a shift of the coordinate $z \rightarrow z + z_0$. Hence, we can introduce an additional condition, fixing the origin of coordinates

$$v(0) = 0 \quad (1.16)$$

The solution for weak shock waves. For weak shock waves $\varepsilon \leq 1$. The solution of system (1.11)–(1.13) will be sought in the form of power series in ε

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots \quad (1.17)$$

and similarly for τ and f .

We will introduce the scale transformation

$$z = \tilde{z}/\varepsilon \quad (1.18)$$

since differentiation with respect to z increases the order of smallness with respect to ε by unity, and we will also take into account the fact that

$$A = \frac{\gamma^2}{2(\gamma^2 - 1)} - \varepsilon^2 \frac{\gamma + 1}{8(\gamma - 1)} \quad (1.19)$$

We substitute expressions (1.17)–(1.19) into Eqs (1.11)–(1.13) and equate terms of like powers of ε . The approximation of ε^0 leads to identities. In the approximation of ε we obtain

$$f_0 = 0, \quad \tau_0 = -v_0 \quad (1.20)$$

but, to determine the explicit form of v_0 we need to consider the next approximation with respect to ε , in which we obtain

$$bdv_0 / d\bar{z} = -(1 - v_0^2) \tag{1.21}$$

$$b = \frac{4\gamma}{(\gamma + 1)^2} \left[\frac{4}{3} \mu^- + \frac{(\gamma - 1)^2}{\gamma} \kappa^- + (\gamma + 1) \alpha^2 D^- \right], \quad \alpha = \alpha_p^- + \frac{\gamma - 1}{\gamma} \alpha_T^-$$

$$f_1 = -\frac{\gamma + 1}{2\gamma} \alpha_{y_1 y_2} D^- \frac{dv_0}{d\bar{z}} \tag{1.22}$$

$$\tau_1 + \nu_1 = \frac{\gamma + 1}{4\gamma} (1 - v_0^2) + \left(\frac{4}{3} \mu^- - \frac{\gamma - 1}{\gamma} \alpha D^- - \frac{\gamma - 1}{\gamma} \kappa^- \right) \frac{dv_0}{d\bar{z}} \tag{1.23}$$

The solution of Eq. (1.21) which satisfies the conditions

$$v_0(\mp\infty) = \pm 1, \quad v_0(0) = 0$$

is

$$v_0 = -\text{th}(z/b) \tag{1.24}$$

It then follows from (1.22) that

$$f_1 = \frac{(\gamma + 1)^2}{2\gamma} \frac{D^-}{b} \alpha_{y_1 y_2} \text{ch}^{-2} \left(\frac{\bar{z}}{b} \right) \tag{1.25}$$

Hence, in complete agreement with well-known results [2], the change in the concentration is a second-order infinitesimal in ϵ , and within the shock wave the concentration of the light component increases, reaching a maximum at the point $z = 0$ (the centre of the wave), and then decreases to an asymptotic value. It also follows from (1.25) that the diffusion rate of the heavy molecules of the mixture is everywhere greater than that of the light molecules.

To determine ν_1 and τ_1 we must change to the third approximation in ϵ . In this approximation it is necessary to take into account the Burnett terms in the equations and the changes in the Navier-Stokes transfer coefficients due to changes in the temperature and pressure

$$\mu = \mu^- + \epsilon \frac{\gamma - 1}{2(\gamma + 1)} \tau_0 \left(\frac{d\mu}{dT} \right)^- + \dots \tag{1.26}$$

and also the expressions for κ , ρD , α_T .

Here we will only derive those necessary for further correction, taking into account the fact that $D \sim p^{-1}$ and the concentrations can be assumed constant.

The structure of the equations is the same as that of the equations of the previous approximation: the first-order differential equation for ν_1 and the relations which define f_2 and $\tau_2 + \nu_2$. The equation for ν_1 has the form

$$\frac{d\nu_1}{d\bar{z}} - 2\nu_1 \nu_1 = \frac{4\gamma}{(\gamma + 1)^3} \frac{G}{b^2} \frac{d^2 v_0}{d\bar{z}^2} \tag{1.27}$$

For what follows it will be convenient to split the coefficient G into three parts

$$G = G_1 + G_2 + G_3 \tag{1.28}$$

$$\begin{aligned} G_1 = & -\frac{16}{9} \mu^2 - \frac{(3\gamma + 1)(\gamma - 1)^3}{\gamma^2} \kappa^2 - \frac{4}{3} (\gamma - 1)(3 - \gamma) \mu \kappa - \\ & - \frac{4}{3} \frac{(\gamma + 1)^2}{\gamma} y_1 y_2 \left(\alpha_p^2 + \frac{\gamma - 1}{\gamma} \alpha_T^2 + \frac{3\gamma^2 + \gamma - 1}{\gamma(\gamma + 1)} \alpha_p \alpha_T \right) \mu D - \frac{\gamma^2 - 1}{\gamma^2} \left[(\gamma^3 + \right. \\ & \left. + 2\gamma^2 - 1) \alpha_p^2 + (\gamma^2 + 2\gamma - 3) \alpha_T^2 + \frac{\gamma - 1}{\gamma^2} (2\gamma^3 + 7\gamma^2 + 3\gamma - 1) \alpha_p \alpha_T \right] y_1 y_2 \kappa D - \end{aligned} \tag{1.29}$$

$$\begin{aligned}
 & -(\gamma+1)^2 \left[\frac{\gamma+1}{\gamma} + y_1 y_2 \left(\frac{\gamma^2 + \gamma + 1}{\gamma} \alpha_p^2 + \frac{2(\gamma-1)(\gamma^2+1)}{\gamma^2} \alpha_T^2 + \right. \right. \\
 & \left. \left. + \frac{2\gamma^3 + \gamma^2 + \gamma - 1}{\gamma^2} \alpha_p \alpha_T \right) \right] y_1 y_2 \alpha^2 D^2 \\
 G_2 = & \frac{\gamma^2 - 1}{4} b \left[\left(\frac{d\mu}{dT} \right)^- \frac{4}{3} + \frac{(\gamma-1)^2}{\gamma} \left(\frac{d\kappa}{dT} \right)^- + y_1 y_2 \alpha_T \left(\frac{\gamma-1}{\gamma} \alpha + \alpha_p \right) \left(\frac{dD}{dT} \right)^- + \right. \\
 & \left. + 2 \frac{\gamma-1}{\gamma} y_1 y_2 \alpha D \left(\frac{d\alpha_T}{dT} \right)^- \right] \tag{1.30}
 \end{aligned}$$

$$G_3 = \tilde{\omega}_1 \left(\gamma + \frac{(\gamma-1)(\gamma+1)^3}{\gamma^2} \right) \mu^2 - \tilde{\omega}_3 \frac{(\gamma+1)^2(\gamma-1)}{\gamma} \left(1 + \frac{\gamma^3}{(\gamma+1)^3} \right) \mu \kappa + \tilde{\omega}_5 \gamma^2 \frac{\gamma-1}{\gamma+1} \kappa^2 \tag{1.31}$$

where G_1 is the contribution from the main (Navier–Stokes) terms in the hydrodynamic equations with unperturbed values of the kinetic coefficients, G_2 is the additional contribution from them due to changes in the kinetic coefficients and G_3 is the contribution from the Burnett terms.

The solution of Eq. (1.27) which satisfies the boundary conditions

$$v_0(\mp\infty) = 0, \quad v_1(0) = 0$$

is

$$v_1(\bar{z}) = \frac{8\gamma}{(\gamma+1)^2} \frac{G}{b^2} \frac{\ln(\text{ch}\bar{z})}{\text{ch}^2\bar{z}} \tag{1.32}$$

The width of the shock wave Λ is found from the maximum slope of the density profile of the mixture using the relation

$$\Lambda^{-1} = (\rho^+ - \rho^-)^{-1} \left| \frac{d\rho}{dz} \right|_{\max} \tag{1.33}$$

The terms of the expansion of the density in powers of ε are obtained from (1.10)

$$\begin{aligned}
 \rho = & \frac{\gamma+1}{\gamma} + \varepsilon \varphi_0 + \varepsilon^2 \varphi_1 + \dots \\
 \varphi_0 = & -\frac{\gamma+1}{\gamma} \frac{v_0}{2}, \quad \varphi_1 = \frac{1}{2} \left(\frac{\gamma+1}{\gamma} \right)^2 \left(\frac{\gamma+1}{2\gamma} v_0 - v_1 \right)
 \end{aligned} \tag{1.34}$$

Then Λ^{-1} can also be written in the form of a series in ε

$$\Lambda^{-1} = \varepsilon \Lambda_1^{-1} + \varepsilon^2 \Lambda_2^{-1} + \dots \tag{1.35}$$

(the expansion begins with the first power of ε , since when $\varepsilon = 0$ the shock wave disappears, i.e. $\Lambda \rightarrow \infty$).

In the main approximation in ε from (1.33)–(1.35) we obtain the well-known formula for the width of the shock wave [2]

$$\Lambda_1^{-1} = \varepsilon (\rho^+ - \rho^-)^{-1} \left| \frac{d\varphi_0}{dz} \right|_{z=0} = \frac{\varepsilon}{2b} \tag{1.36}$$

We will calculate the correction to Λ_1 , which arises when φ_1 is taken into account in the expansion of the density (1.34). The point $z = z^*$, at which the density profile has the maximum slope, is found from the condition

$$\frac{d^2}{dz^2}(\epsilon\varphi_0 + \epsilon^2\varphi_1) = 0$$

The approximate solution of this equation has the form

$$z^* = \epsilon \left(\frac{\gamma+1}{2\gamma} - \frac{4\gamma}{(\gamma+1)^2} \frac{G}{b^2} \right) + O(\epsilon^2)$$

Then

$$\Lambda^{-1} = (\rho^+ - \rho^-)^{-1} \left| \frac{d(\epsilon\varphi_0 + \epsilon^2\varphi_1)}{dz} \right|_{z=z^*} = \frac{\epsilon}{2b} + O(\epsilon^3) \tag{1.37}$$

Hence, the second approximation makes no contribution to the width of the shock wave. This can obviously be explained by the fact that formula (1.36) agrees satisfactorily with experimental data and the results of numerical calculations up to the values of the Mach number $M \approx 1.8$ (for which $\epsilon \approx 0.437$) [8]. The non-zero correction $\sim \epsilon^2$ obtained in [9] can be explained by the fact that in [9] the point $z = 0$ is taken instead of z^* as the point of maximum slope of the density profile.

We will consider the asymmetry of the profiles of the hydrodynamic quantities and the distances between their centres for the case of single-component gases.

For the quantitative characteristic of the degree of asymmetry of the profiles of the hydrodynamic quantities in the shock wave we will use the asymmetry parameter of the density profile (the asymmetry quotient) [3]

$$Q_\rho = \left(\int_{-\infty}^{z_p^*} \rho_*(z) dz \right) \left(\int_{z_p^*}^{\infty} [1 - \rho_*(z)] dz \right)^{-1}, \quad \rho_*(z) = \frac{\rho(z) - \rho^-}{\rho^+ - \rho^-} \tag{1.38}$$

where z_p^* is the centre of the density profile, i.e.

$$\rho_*(z_p^*) = 1/2 \tag{1.39}$$

The approximate solution of Eq. (1.39) is

$$z_p^* = \epsilon \frac{\gamma+1}{2\gamma} b + O(\epsilon^2) \tag{1.40}$$

Substituting (1.40) into (1.38) we obtain

$$Q_\rho = \left[1 - \epsilon \frac{8\gamma}{(\gamma+1)^2} \frac{G}{b^2} \left(\frac{1}{\ln 2} - 1 \right) \right] \left[1 + \epsilon \frac{8\gamma}{(\gamma+1)^2} \frac{G}{b^2} \left(\frac{1}{\ln 2} - 1 \right) \right]^{-1} \tag{1.41}$$

In the first approximation, the density, temperature and velocity profiles are described by a hyperbolic tangent. They are strictly antisymmetric. Here $Q_i = 1$ ($i = \rho, u, T$). In the next approximation, the profiles become asymmetric. It follows from (1.41) that $Q_\rho > 1$ if $G < 0$ and $Q_\rho < 1$ if $G > 0$.

We need to establish the sign of the coefficient G . The coefficients G_1, G_2 and G_3 can be reduced to the form

$$\begin{aligned} G_1 &= -\frac{(\gamma-1)^3}{\gamma} \left(x - \frac{2}{3}\mu \right)^2 - \left(1 - \frac{(\gamma-1)^3}{4\gamma} \right) \left(\frac{4}{3}\mu \right)^2 < 0 \\ G_2 &= \frac{\gamma(\gamma-1)}{(\gamma+1)^2} \left(\frac{4}{3}\mu + \frac{(\gamma-1)^2}{\gamma} x \right) \left(\frac{4}{3} \frac{d\mu}{dT} + \frac{(\gamma-1)^2}{\gamma} \frac{dx}{dT} \right) > 0 \\ G_3 &= \tilde{\omega}_1 \frac{\gamma^3 + (\gamma-1)(\gamma+1)^3}{\gamma^2} \mu^2 - \tilde{\omega}_3 \frac{(\gamma+1)^2(\gamma-1)}{\gamma} \left(1 + \frac{\gamma^3}{(\gamma+1)^3} \right) \mu x + \tilde{\omega}_5 \gamma^2 \frac{\gamma-1}{\gamma+1} x^2 \end{aligned} \tag{1.42}$$

It can be verified that the contribution to G from the Navier–Stokes terms ($G_1 + G_2$) < 0 . This is

easily done for the power potentials of intermolecular interaction, for which $\kappa \sim \mu \sim T^s$. The limiting cases here are Maxwellian molecules ($s = 1$) and solid elastic spheres ($s = 1/2$). Taking into account the fact that $\kappa = 15 \mu/4$, $\gamma = 5/3$, from (1.29), (1.30) and (1.31) we obtain

$$G_1 = -2.4948 \mu^2, \quad G_2 = 0.1215 s\mu^2, \quad G_3 = 27.2438 \mu^2$$

We have

$$G = G_1 + G_2 + G_3 = 24.8705 \mu^2 > 0$$

for Maxwellian molecules (for solid spheres the value of G is even greater).

Experiments [3] and numerical solutions of Burnett's equations [4] indicate that $Q_p < 1$ when $M \leq 1.8$. At the same time, a numerical solution of the Navier–Stokes equations [4] leads to the qualitatively incorrect result $Q_p > 1$ in this range of Mach numbers.

Hence, only when the Burnett terms are taken into account in the hydrodynamic equations can one obtain qualitatively correct values of the asymmetry parameter of the density profile in a weak shock wave.

The calculation of the centre z_T^* and the asymmetry parameter Q_T of the temperature profile $T_*(z)$ is similar to the calculation of z_p^* and Q_p . It leads to the results

$$Q_T = Q_p, \quad z_T^* = -\varepsilon B + O(\varepsilon^2), \quad B = \frac{1}{2} \frac{\gamma+1}{\gamma-1} - \frac{4}{3} \frac{\gamma}{\gamma-1} \frac{\mu}{b} \quad (1.43)$$

It follows from (1.40) and (1.43) that $z_p^* > 0$ and $z_T^* < 0$, i.e. the temperature profile leads the density profile. This result is physically obvious. Molecules having a higher velocity penetrate into the upstream region of the shock-wave front. The contribution of these molecules to the density is small in view of the small number of such molecules, but the contribution to the temperature is finite in view of their high energy. The effect noted above also occurs in a shock wave of medium intensity.

For the velocity profile, calculation leads to the result $z_u^* = 0$. The velocity profile lies between the temperature and density profiles.

Note that, in the first approximation, the centres of the profiles of the hydrodynamic quantities coincide.

2. THE PROPAGATION OF SMALL-AMPLITUDE PERTURBATIONS IN GASEOUS MIXTURES

The initial problem. We will take as the macroscopic quantities which describe the state of the binary gaseous mixture the mass concentration of one of the components of the mixture $c_1 = \rho_1/\rho$, the hydrodynamic velocity u , the pressure p and the temperature T . Suppose small fluctuations in the macroscopic quantities

$$c_1 = c_{10} + c_1', \quad u = 0 + u', \quad p = p_0 + p', \quad T = T_0 + T' \quad (2.1)$$

are produced in a mixture at rest and in thermodynamic equilibrium. The density fluctuations are defined in terms of these using the equation of state

$$\rho' = \frac{\rho_0}{\rho_0} p' - \frac{\rho_0}{T_0} T' - \frac{\rho_0^2 T_0}{\rho_0} (R_1 - R_2) c_1' \quad (2.2)$$

Substituting (2.1) and (2.2) into the one-dimensional Burnett equations [5] and linearizing, we obtain

$$\begin{aligned} \frac{1}{\rho_0} \frac{\partial p'}{\partial t} - \frac{1}{T_0} \frac{\partial T'}{\partial t} - \frac{\rho_0 T_0}{\rho_0} \frac{\partial c_1'}{\partial t} + \frac{\partial u'}{\partial z} &= 0 \\ \frac{\partial c_1'}{\partial t} - D \left[\frac{\partial^2 c_1'}{\partial z^2} + c_{10} c_{20} \left(\frac{\alpha_p}{\rho_0} \frac{\partial^2 p'}{\partial z^2} + \frac{\alpha_T}{T_0} \frac{\partial^2 T'}{\partial z^2} \right) \right] + \frac{\beta_4}{\rho_0} \frac{\partial^3 u'}{\partial z^3} &= 0 \\ \rho_0 \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial z} - \frac{4}{3} \mu \frac{\partial^2 u'}{\partial z^2} - \beta_1 \frac{\partial^3 p'}{\partial z^3} - \beta_2 \frac{\partial^3 T'}{\partial z^3} - \beta_3 \frac{\partial^3 c_1'}{\partial z^3} &= 0 \\ \frac{3}{2} \frac{\rho_0}{T_0} \frac{\partial T'}{\partial t} + \rho_0 \frac{\partial u'}{\partial z} - \kappa \frac{\partial^2 T'}{\partial z^2} - \frac{5}{2} \frac{\rho_0 \rho_0^2 \alpha}{n_0^2 m_1 m_2} D \left[\frac{\partial^2 c_1'}{\partial z^2} + \right. & \end{aligned}$$

$$+c_{10}c_{20}\left(\frac{\alpha_p}{\rho_0}\frac{\partial^2 p'}{\partial z^2}+\frac{\alpha_T}{T_0}\frac{\partial^2 T'}{\partial z^2}\right)-\left(\beta_5+\beta_4\frac{5}{2}\frac{\rho_0\rho_0\alpha}{n_0^2m_1m_2}\right)\frac{\partial^3 u}{\partial z^3}=0 \tag{2.3}$$

We will seek the solution of system (2.3) in the form of plane waves

$$\Gamma'(t, z) = \hat{\Gamma} \exp(ikz - i\omega t), \quad \Gamma' = (c'_1, u', p', T') \tag{2.4}$$

where k is the wave number and ω is the frequency. Substituting (2.4) into (2.3) we obtain a system of linear homogeneous algebraic equations for the amplitudes

$$\begin{aligned} i\omega\frac{\rho_0T_0}{\rho_0}(R_1 - R_2)\hat{c}_1 + ik\hat{u} - i\omega\frac{1}{\rho_0}\hat{p} + i\omega\frac{1}{T_0}\hat{T} &= 0 \\ (-i\omega + k^2D)\hat{c}_1 - ik^3\beta_4\frac{1}{\rho_0}\hat{u} + k^2c_{10}c_{20}D\left(\frac{\alpha_p}{\rho_0}\hat{p} + \frac{\alpha_T}{T_0}\hat{T}\right) &= 0 \\ ik^3\beta_3\hat{c}_1 - \left(i\omega\rho_0 - k^2\frac{4}{3}\mu\right)\hat{u} + i(k + k^3\beta_1)\hat{p} + ik^3\beta_2\hat{T} &= 0 \\ k^2\frac{5}{2}\frac{\rho_0\rho_0^2\alpha}{n_0^2m_1m_2}D\hat{c}_1 + \left[ik\rho_0 + ik^3\left(\beta_5 + \beta_4\frac{5}{2}\frac{\rho_0\rho_0\alpha}{n_0^2m_1m_2}\right) \right]\hat{u} + \\ + k^2\frac{5}{2}\frac{\rho_0^2\alpha}{n_0^2m_1m_2}c_{10}c_{20}D\hat{p} - \left[i\omega\frac{3}{2}\frac{\rho_0}{T_0} - k^2\left(\kappa + \frac{5}{2}y_{10}y_{20}\frac{\alpha\rho_0}{T_0}D\right) \right]\hat{T} &= 0 \end{aligned} \tag{2.5}$$

Here $\alpha = \alpha_p + 2\alpha_T/5$ and we have omitted the zero subscripts for simplicity.

For non-trivial solutions of (2.5) to exist it is necessary for the determinant of its coefficients to be zero. Expanding the determinant, we obtain the dispersion equation

$$\begin{aligned} (\omega^2 - ka_0^2 - i\omega k^2b_*)[\omega^2 - i\omega k^2(\chi + D^*)] - \omega^2k^4\left[\left(\frac{2}{3}\chi + \frac{5}{3}y_1y_2\alpha^2D\right)\left(\frac{4}{3}v + \chi - D^*\right) \right. \\ \left. + \frac{5}{3}(1 + y_1y_2\alpha_p^2)\chi D\right] + k^6\frac{5}{3}\chi D\left[\frac{3}{5}a_0^2 - i\omega(1 + y_1y_2\alpha_p^2)\frac{4}{3}v\right] + \\ + \omega k^4\frac{2}{3}\frac{T}{\rho}\left[\frac{5}{2}\frac{p}{T}\beta_1 + \beta_2 + (R_1 - R_2)(4\alpha_p + \alpha_T)\beta_4 + \frac{1}{T}\beta_5\right] = 0 \end{aligned} \tag{2.6}$$

Here

$$\begin{aligned} b_* = \frac{16}{15}b = \frac{4}{3}v + \frac{2}{3}\chi + \frac{5}{3}y_1y_2\alpha^2D, \quad v = \frac{\mu}{\rho}, \quad \chi = \frac{\kappa}{\rho c_p} \\ a_0 = \left(\frac{5}{3}\frac{p}{\rho}\right)^{1/2}, \quad D^* = D\left(1 + \frac{2}{5}y_1y_2\alpha_T^2\right) \end{aligned} \tag{2.7}$$

v, χ are the kinematic viscosity and the thermal diffusivity, u_0 is the adiabatic velocity of sound, and the coefficient b_* occurs in the Kirchhoff–Stokes formula as a factor (it was given previously in Eq. (1.21) in dimensionless form). When obtaining Eq. (2.6) we dropped terms of the order of Kn^3 since we are considering the Burnett approximation.

Equation (2.6) is a fourth-order algebraic equation in ω , if k is regarded as a given quantity (the initial problem). The roots of the equation will be sought in the form of an expansion in k . In fact, the expansion is carried out in terms of dimensionless quantities $vk/a_0, \chi k/a_0, Dk/a_0$. Taking into account the fact that $v \sim \chi \sim D \sim \bar{l} \bar{v}, k = 2\pi/\lambda$ (\bar{l} is the mean free path of the molecules, \bar{v} is the thermal velocity of the molecules, $\bar{v} \sim a_0$ and λ is the wavelength), we obtain that these quantities are proportional to the Knudsen number $Kn = \bar{l}/\lambda$.

Hence we obtain two pure imaginary roots

$$\begin{aligned}\omega_{1,2} &= -ik^2\theta + O(k^4) \\ \theta &= \frac{1}{2}(\chi + D^*) \pm \frac{1}{2}\sqrt{(\chi + D^*)^2 - 4\chi D}\end{aligned}\quad (2.8)$$

Another pair of roots has the form

$$\begin{aligned}\omega_{3,4} &= \pm a_0 k - i\frac{1}{2}b_* k^2 \mp \frac{h_1 - h_2}{2a_0} k^3 + O(k^4) \\ h_1 &= \frac{4}{9}v^2 + \frac{7}{9}\chi^2 - \frac{4}{9}v\chi + y_1 y_2 \alpha D \left[-\frac{10}{9}\alpha v + \frac{5}{9}\left(\alpha + \frac{12}{5}\alpha_T\right)\chi + \right. \\ &\quad \left. + \frac{5}{3}\alpha\left(1 + \frac{5}{12}y_1 y_2\left(\alpha^2 + \frac{24}{25}\alpha_T^2\right)\right)D\right] \\ h_2 &= \frac{5}{3}\tilde{\omega}_1 v^2 + \frac{25}{6}\tilde{\omega}_5 \chi^2 - \frac{10}{3}\tilde{\omega}_3 v\chi - \frac{5}{3}\tilde{\omega}_2 y_1 y_2 \alpha_p v D + c_1 c_2 \left(\frac{25}{9}\alpha_p + \frac{20}{9}\alpha_T\right)\alpha_p D^2\end{aligned}\quad (2.9)$$

Here we have denoted by h_1 and h_2 the contributions which derive from the Navier–Stokes and Burnett approximations.

A definite mode corresponds to each root. We will explain the physical meaning of these modes. We substitute into (2.5) the first root $\omega_1 \approx -ik^2\chi/2$, in which we have neglected the small term proportional to α_T . Solving (2.5) for the amplitudes $\hat{\Gamma}$, we obtain that

$$\hat{u} \sim \hat{p} \sim \hat{c}_1 \sim \text{Kn} \hat{T} \ll \hat{T}$$

In this mode the temperature fluctuations decay due to the thermal conductivity when $u = 0$ and p and c_1 are constant (the thermal mode). Similarly, the second root $\omega_2 \approx -ik^2 D/2$ leads to the relation

$$\hat{u} \sim \hat{p} \sim \hat{T} \sim \text{Kn} c_1 \ll c_1$$

Here the concentration fluctuations decay due to diffusion when $u = 0$ and p and T are constant (the diffusion mode). These modes are weakly related to one another via the thermal diffusion. Substituting ω_3 and ω_4 into (2.5) we obtain that the motion of the medium consists of two sound waves propagating in opposite directions. The number of independent modes of motion, having different physical meanings, is identical with the number of conservation laws (the hydrodynamic equations).

The boundary-value problem. For comparison with experimental results obtained by measurements of the velocity and absorption of coefficient of sound, in which the frequency ω is a specified real quantity, we need to solve dispersion equation (2.6) for the wave number k (the boundary-value problem). The order of Eq. (2.6) is higher with respect to k than with respect to ω , so there are additional roots. Numerical solution of the dispersion equation in the case of a single-component gas [10] indicates that the wave corresponding to these roots is strongly attenuated. The relation $\text{Im} k_1 \ll \text{Im} k_2 \ll \text{Im} k_3$ holds between the attenuation factors of the acoustic, thermal and additional waves (the I-wave, the II-wave and the III-wave in the terminology of [10]) in the low-frequency domains. Moreover, the III-wave possesses non-physical properties: $\text{Im} k_3 \gg \text{Re} k_3$ unlike the I-wave ($\text{Im} k_1 \ll \text{Re} k_1$) and the II-wave ($\text{Im} k_2 \ll \text{Re} k_2$), and at high frequencies $\text{Re} k_3$ changes sign (this denotes that the direction of the wave propagation is reversed). Some doubt has been expressed regarding its reality [10].

From a mathematical point of view it is much simpler to convert expansions (2.8) and (2.9) rather than obtain solutions of Eq. (2.6) in the form of series in ω . Moreover, in this case we get rid of non-physical roots. We then have from (2.9)

$$\begin{aligned}k_{3,4} &= \pm \frac{1}{a_0} \omega \pm i \frac{b_*}{2a_0^3} \omega^2 \mp \frac{d}{2a_0^5} \omega^3 + O(\omega^4) \\ d &= \frac{4}{3}v^2 - \frac{1}{3}\chi^2 + \frac{20}{9}v\chi + y_1 y_2 \alpha D \left[\frac{50}{9}\alpha v + \right. \\ &\quad \left. + \left(\frac{5}{3}\alpha - \frac{4}{3}\alpha_T\right)\chi - \frac{5}{3}\alpha D \left(1 - y_1 y_2 \left(\frac{5}{4}\alpha^2 - \frac{2}{5}\alpha_T^2\right)\right)\right] + h_2\end{aligned}\quad (2.10)$$

for the acoustic modes. For the thermal and diffusion modes we obtain from (2.8)

$$k_{1,2} = \pm(1+i)\sqrt{\frac{\omega}{2\theta}} + O(\omega^{3/2}) \quad (2.11)$$

From (2.10) we obtain the following equations for the phase velocity of sound v_{ph} and the absorption coefficient at the wavelength S

$$\frac{a_0}{v_{ph}} = \operatorname{Re}\left(\frac{a_0 k}{\omega}\right) = \pm\left(1 - \frac{d}{2a_0^4}\omega^2 + \dots\right) \quad (2.12)$$

$$S = \operatorname{Im}\left(\frac{a_0 k}{\omega}\right) = \pm\frac{b\omega}{2a_0^2} + \dots \quad (2.13)$$

It follows from (2.12) that there is a dispersion of the sound curves, the dispersion coefficient $d \sim \operatorname{Kn}^2$ and it contains both the Navier–Stokes contributions and the Burnett contributions, and they are of the same order of magnitude. An analysis of the expression for d shows that in general, $d > 0$. Consequently, the velocity of sound increases as the frequency increases (positive dispersion). The attenuation factor is determined solely by the Navier–Stokes terms, while the next term gives the contribution of the super-Burnett approximation also.

Another situation arises in a mixture in which the masses of the molecules differ considerably ($m_2 \gg m_1$). In a helium–xenon mixture with an 80% xenon content it has been shown experimentally that there is negative dispersion. We can assume here that $\sqrt{m_1/m_2} \sim y_1 \ll 1$. Then, we can obtain from the general formulae [5, 7]

$$\mu_1 \approx \mu_{11}, \quad \mu_2 \approx \frac{y_2}{y_1} \frac{k_B \theta T}{2A_{12}} \frac{1}{\sqrt{m_2} \Omega_{12}^{(1,1)}} \ll \mu_1$$

$$\kappa_1 \approx \kappa_{11}, \quad \kappa_2 \approx \frac{y_2}{y_1} \kappa_{22} \frac{\Omega_2^{(2,2)}}{5\Omega_{12}^{(1,1)}} - \kappa_1$$

$$D = \frac{3}{16} \frac{k_B \theta T}{m_2 n} \frac{1}{\Omega_{12}^{(1,1)}}$$

Consequently

$$v \sim \chi \sim \sqrt{m_2 m_1} D \ll D$$

Hence the term with D^2 is the largest in the dispersion coefficient. As estimates show, the Burnett contributions are small. We then have

$$d = -\frac{5}{3} \frac{y_2}{y_1} D^2 < 0$$

which leads to negative dispersion.

One further fact is worth mentioning. The domain of applicability of the solution in the form of a series in ω is limited by the frequency $\omega^* \sim (m_1/m_2)\tau^{-1}$ (τ^{-1} is the collision frequency of the light molecules), for which the second term in (2.12) becomes of the same order as the first. The limiting frequency ω^* is identical with the well-known estimate of the inverse relaxation time of the temperature of the components of the mixture. In the frequency range $\omega \geq \omega^*$ it is necessary to use the equations of two-temperature gas dynamics or the equations of two-velocity and two-temperature gas dynamics.

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